

Artificial Intelligence

AI: Search: informed, uninformed, adversarial; logic, propositional, predicate; reasoning under uncertainty topics — conditional independence representation, exact inference through variable elimination, and approximate inference through sampling.

Mathematic Logic

- Language for mathematics
- Precise & ambiguous meaning
- Natural languages are ambiguous.

→ Basis of all mathematical reasoning & automatic reasoning.

- ① Propositional logic
 - ② First order logic
 - ③ Second order logic
 - ④ Fussy logic
- } gate syllabus → Predicate logic

Propositional logic

→ Every variable is either true or false { eg: p is a variable → $\begin{matrix} T & F \\ F & 0 \end{matrix}$ } values
↑
propositional variable
boolean variable

→ Proposition is a declarative sentence that declares a fact which can be either true or false
aka (statement) (assertive)

- eg: - Jaipur is the capital of India — F } both are proposition
• Jaipur is the capital of Rajasthan — T
• All cows are brown — F
• $2 \times 2 = 5$ — F
• Earth is farther from sun than earth } value need not be known if T or F
• There are 1000 stones on Mars } we just need to be sure the answer is binary

Counter example

→ Command is not a proposition
imperative

→ question is not a proposition
interrogative

→ $x+2=2x$ } is not a proposition
→ $4x+2=3n$ }

→ T or false depends on value of x
→ free variable which can be any integer

T or F → depends on value of x
→ Value of x is not a fact is an issue
→ Value of x unknown is not an issue

For all integer n , $2n$ is an even number → proposition

→ even though there is variable
→ The Truth or false of statement doesn't depend on value of variable

→ S : "S is false" → Not a propositional logic statement
paradoxical statements are not propositional statements

propositional variable

Variable name : proposition statement

eg: A : The sun is shining } C : if A is true then B is true
 B : I feel happy }

→ if S is false we say **truth value** of S is false

Atomic proposition → Does not depend on truth value of other propositions
→ Cannot be broken

eg:- It is a prime number

Compound eg:- It is a prime number & new delhi is the capital of India → Proposition but not atomic

→ Every propositional variable is a atomic proposition

Compound proposition → created from atomic proposition with connectives (logic operator)

P : 4 is a prime number.

Q : New Delhi is the capital of India.

} Atomic } Atomic

4 is a prime number, AND New Delhi is the capital of India. → $P \wedge Q$

4 is NOT a prime number. → $\text{not } P$

4 is a prime number, OR New Delhi is the capital of India. → $P \vee Q$

4 is a prime number, OR New Delhi is the capital of India, BUT NOT BOTH. → $P \oplus Q$

IF 4 is a prime number, THEN New Delhi is the capital of India. → $P \Rightarrow Q$

} Compound.

STANDARD LOGICAL CONNECTIVES (OPERATORS)

1. NOT unary

2. AND

3. OR

4. Exclusive OR

5. Implication

6. Bi-Implication

7. NAND

8. NOR

} Binary

- 1) NOT $\neg P, \sim P, \bar{P}, P', \text{not } P$
- 2) AND
 Conjunction $P \wedge Q$ English words $\rightarrow$ even though, and, however, yet, although, though, but, nevertheless, moreover
- 3) OR
 Disjunction $P \vee Q$ Inclusive OR
- 4) XOR
 Exclusive OR $P \oplus Q$
- 5) NAND
 not AND $P \uparrow Q, \overline{P \wedge Q}, \neg(P \wedge Q)$
- 6) NOR
 not OR $P \downarrow Q, \overline{P \vee Q}, \neg(P \vee Q)$
- 7) Implication $P \rightarrow Q, Q \leftarrow P$
 $\rightarrow$ condition
 $\rightarrow$ implication

ex: If today is Sunday then I am happy

$\rightarrow$ Condition
 $\rightarrow$ Antecedent
 $\rightarrow$ Hypothesis
 $\rightarrow$ premise

$\rightarrow$ consequence
 $\rightarrow$ conclusion

$\rightarrow$ Result of implication $P \rightarrow Q$ is T if the implication holds for values of P and Q otherwise it is false

P	Q	$P \rightarrow Q$
0	0	T
0	1	T
1	0	F
1	1	T

hypothesis — T
 conclusion — F

implication doesn't hold

$$\rightarrow P \rightarrow Q \neq Q \rightarrow P$$

$$\rightarrow \text{Equivalent statement } P \rightarrow Q \equiv (\neg P \vee Q)$$

$$\rightarrow \text{Contrapositive } P \rightarrow Q \equiv \neg Q \rightarrow \neg P$$

Sidequest

Necessary & Sufficient conditions ☆ α is sufficient for $\beta \equiv \beta$ is necessary for α

$\triangleright$ if P is necessary for Q $\mid$ eg: filling gate application is necessary for ranking gate
 then $\neg P \rightarrow \neg Q$
 $\Rightarrow Q \rightarrow P$

P
 $\neg P \rightarrow \neg Q$
 $Q \rightarrow P$

$\triangleright$ if P is sufficient for Q $\mid$ eg: getting AIR 1 is sufficient for getting admission in IISc
 $P \rightarrow Q$

P $P \rightarrow Q$

▷ if P is necessary & sufficient for Q
then $P \rightarrow Q$
 $Q \rightarrow P$

$P \rightarrow Q$: prop. logic Expression

- ① P Implies Q ⑤ when P , Q
② If P then Q ⑥ Given that P , Q
③ If P , Q ⑦ provided that P , Q
④ whenever P , Q

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$p \rightarrow q$

"if p, then q" "if p, q" "p is sufficient for q" "q if p" "q <u>when</u> p" "a necessary condition for p is q" "<u>q unless $\neg p$</u>"	"p implies q" "p only if q" "a sufficient condition for q is p" "q <u>whenever</u> p" "q is necessary for p" "q follows from p"
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$$p \vee q \equiv q \rightarrow p$$

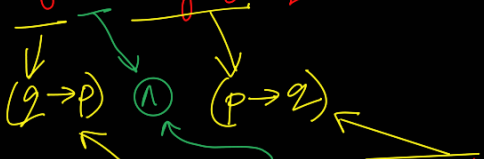
Bi-Implication

$\rightarrow p \rightarrow q \wedge q \rightarrow p$ is bi implication

$$\begin{aligned} \rightarrow P \rightarrow Q &\equiv \bar{P} \vee Q \\ \rightarrow Q \rightarrow P &\equiv \bar{Q} \vee P \end{aligned} \quad \left. \begin{array}{l} \rightarrow P \rightarrow Q \equiv \bar{P} \vee Q \\ \rightarrow Q \rightarrow P \equiv \bar{Q} \vee P \end{array} \right\} P \rightarrow Q \wedge Q \rightarrow P \equiv (\bar{P} \vee Q) \wedge (\bar{Q} \vee P) \\ &\equiv (\bar{P} \wedge \bar{Q}) \vee (\bar{P} \wedge P) \vee (Q \wedge \bar{Q}) \vee (Q \wedge P) \\ &\equiv (\bar{P} \wedge \bar{Q}) \vee (P \wedge Q) \\ &\equiv P \text{ XNOR } Q \equiv \neg(P \oplus Q)$$

English translations

- p if and only if q



- p is necessary and sufficient for q

Definition $\equiv$ Bitmultiplication $\longrightarrow$ eg. Even numbers $\leftrightarrow$ integer multiple of 2

Property $\equiv$ implication $\longleftrightarrow$ eg. even numbers $\rightarrow$ integer
integers $\nrightarrow$ even numbers

Atomic proposition $\rightarrow$ cannot be broken down into simpler proposition } represented by propositional variables
eg: - p, q

Compound proposition $\rightarrow$ Can be broken down further into combination of simpler atomic proposition using connectives } represented by propositional formula
eg. $p \wedge q$, $p \oplus q$

Truth table & precedence of propositional operators

Precedence - Bank Notes Can Discount Income Bureau

Bracket $\{ \}$
 Negation $\neg$
 Conjunction $\wedge$
 Disjunction $\vee$
 Implication $\rightarrow$
 BiImplication $\leftrightarrow$

$\oplus, \uparrow, \downarrow$

$\rightarrow$ precedence not defined
 ☆

• On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.

• Puzzle 1: You meet two people on the island, Arnold and Bob. Arnold says "Either I am a knave, or Bob is a knight". Is Arnold a knight or a knave? What about Bob?

$\therefore$ Finally Arnold is knight & Bob is knight

Arnold is knave = p

Bob is a knight = q

Case 1 $\rightarrow$ Arnold is Knight

$\Rightarrow p \vee q = \text{true}$

p is false $\rightarrow q = \text{true}$

$\therefore$ Bob is knight

Case 2 $\rightarrow$ Arnold is knave

$\Rightarrow p \vee q = \text{false}$

p is true $\Rightarrow$ contradiction

cannot be the case

B is knight = q A (tells)

A & B are opposite types = p B (tells)

In [Sm78] Smullyan posed many puzzles about an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types?"

Truth table

$\neg p \vee q$	A	B	
0 0	knave	knave	$\rightarrow$ no contradiction
0 1	knight	knight	$\rightarrow$ contradiction
1 0	knight	knave	$\rightarrow$ contradiction
1 1	knave	knight	$\rightarrow$ contradiction

both are knaves

Tautology; Contradiction & Contingency

(valid) Tautology $\rightarrow$ Always true (unsatisfiable)

(fallacy) Contradiction $\rightarrow$ Always false (unsatisfiable)

Contingency $\rightarrow$ Sometimes true, Sometimes false

Satisfiable } $\rightarrow$ Can be made true $\rightarrow$ Tautology / Contingency
 Consistent }

always true	sometimes true/sometimes false	always false
tautology	contingency	contradiction
logically true	contingency	logically false
valid	invalid	
unfalsifiable	falsifiable	
	satisfiable	unsatisfiable

By case Method

→ Alternate to truth table method for solving questions.

3. (9 points) Which of the following compound propositions are tautologies, which are contradictions, and which are neither (contingencies). Justify your answer for each.

a. $\neg q \vee (p \wedge q)$

Case $q = 0$

$$\text{Let } k = \neg q \vee (p \wedge q)$$

$$k = 1$$

$\Rightarrow k \rightarrow$ not a contradiction

Case $q = 1$

$$k = 0 \vee (p \wedge 1)$$

$$k = p$$

Logical Equivalence (Symbol $\equiv$) some authors ($\Leftrightarrow$)

eg:- It is not true that John is both intelligent & hardworking $\neg(I \wedge H)$
 $\equiv$ John is not intelligent OR John is not hardworking $\equiv (\neg I \vee \neg H)$

★ α & β are equivalent if $\alpha \leftrightarrow \beta$ is a tautology
 then we say $\alpha \equiv \beta$ "equivalent to"

Part II: Proving logical equivalence using laws of propositional logic (50 pt.)

4. (30 pt.) Use the laws of propositional logic to prove that the following compound propositions are logically equivalent.

- (5 pt.) $(p \vee q) \wedge \neg(p \wedge \neg q)$ and q
- (5 pt.) $\neg p \rightarrow \neg(q \vee r)$ and $(q \rightarrow p) \wedge (r \rightarrow p)$
- (10 pt.) $\neg(p \vee (q \wedge (\neg p \rightarrow \neg r)))$ and $\neg p \wedge (q \rightarrow r)$
- (10 pt.) $p \leftrightarrow (p \wedge q)$ and $\neg p \vee q$

$$b) \neg p \rightarrow \neg(q \vee r) \equiv p \vee \neg(q \vee r)$$

$$(q \rightarrow p) \wedge (r \rightarrow p) \equiv (\neg q \vee p) \wedge (\neg r \vee p)$$

$$\equiv (\neg q \wedge \neg r) \vee p \wedge (\neg q \vee \neg r) \vee p$$

$$\equiv (\neg q \wedge \neg r) \vee p$$

$$\equiv \neg(q \vee r) \vee p$$

$$c) \neg(p \vee (q \wedge (\neg p \rightarrow \neg r))) \equiv \neg p \wedge \neg(q \wedge (p \vee \neg r)) \equiv \neg p \wedge (\neg q \vee \neg(p \vee \neg r))$$

$$\equiv \neg p \wedge (\neg q \vee (\neg p \wedge r)) = (\neg p \wedge \neg q) \vee (\neg p \wedge (\neg p \wedge r)) = (\neg p \wedge \neg q) \vee (\neg p \wedge r)$$

$$\equiv \neg p \wedge (\neg q \vee r) \equiv \neg p \wedge (q \rightarrow r)$$

$$a) (p \vee q) \wedge \neg(p \wedge \neg q)$$

$$\equiv (p \vee q) \wedge (\neg p \vee q)$$

$$\equiv \underbrace{(p \wedge \neg p)}_0 \vee (p \wedge q) \vee (q \wedge \neg p) \vee (q \wedge q)$$

$$\equiv q \wedge \underbrace{(p \vee \neg p)}_1 \vee q$$

$$\equiv q$$

$$a) p \leftrightarrow (p \wedge q) \equiv [p \wedge (p \wedge q)] \vee [\neg p \wedge \neg(p \wedge q)] \quad (A \vee \neg B \equiv A \vee \neg B)$$

$$\equiv (p \wedge q) \vee \neg p \wedge (\neg p \vee \neg q)$$

$$\equiv (p \wedge q) \vee \neg p \vee (\neg p \wedge \neg q)$$

$$\equiv p \wedge q \vee \neg p$$

$$\equiv p \wedge q \vee \neg p \vee (\neg p \wedge \neg q)$$

$$\equiv q(\underline{p \vee \neg p}) \vee \neg p$$

$$\equiv \underline{q \vee \neg p}$$

Both of these are $\equiv \neg p$. so no issue

☆ To check whether equivalent, case by case method is best. Apply to above problems and see easily.

Prove that the following expressions are logically equivalent. You can use either a truth table or laws of propositional logic! If you choose to use a truth table, please begin the enumeration of your inputs as F, F, F, instead of T, T, T, as we have discussed in class and discussion. If you choose to use propositional logic laws, you **must** document which law you are using for every derivation.

$$p \Rightarrow (q \vee r), (p \wedge (\neg q)) \Rightarrow r, (p \wedge (\neg r)) \Rightarrow q$$

$$\begin{array}{c} \downarrow \quad \downarrow \quad \downarrow \\ p' + q + r \quad (pq')' + r \quad (pr')' + q \\ \equiv p' + q + r \quad \equiv p' + r + q \end{array}$$

all are equivalent

Converse, Inverse & Contrapositive

$$\alpha \rightarrow \beta \text{ — proposition (implication) } \equiv \neg \beta \rightarrow \neg \alpha \text{ — contrapositive}$$

$$\beta \rightarrow \alpha \text{ — converse } \equiv \neg \alpha \rightarrow \neg \beta \text{ — inverse}$$

eg:- "The home team wins whereas it is raining" $p \rightarrow q$

Converse "If the home team wins, then it is raining"

Inverse "If it is not raining, then the home team won't win"

Contrapositive "If the home team doesn't win, then it is not raining"

English to propositional translation

eg:- You can join the army only if you are 18+. $P \rightarrow Q$

If you are not 18+, then you can't join the army ($\neg 19 \rightarrow \neg 1P$)

Unless in english

$p \text{ unless } q \equiv p \text{ if not } q \equiv \neg q \rightarrow p$

$\mathcal{A}$ unless $\mathcal{B}$ is equivalent to $\mathcal{A}$ if not $\mathcal{B}$ which is symbolized $\sim \mathcal{B} \rightarrow \mathcal{A}$	unless $\mathcal{A}, \mathcal{B}$ is equivalent to if not $\mathcal{A}$, then $\mathcal{B}$ which is symbolized $\sim \mathcal{A} \rightarrow \mathcal{B}$
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eg: - If you don't leave early, you'll miss the train unless train is late. No precedence in English language.

$$p \rightarrow (p \rightarrow q) \equiv p \vee \neg p \vee q$$

Logical Identities

- ① $\neg(\neg P) = P$ Double negation
 - ② $P \vee T = T$
 $P \wedge F = F$ } Domination Law
 - ③ $A \vee \neg A = T$
 $A \wedge \neg A = F$ } I am my worst enemy's best friend Law
 - ④ $P \vee F = P$
 $P \wedge T = P$ } Identity Law
 - ⑤ $a \wedge b = b \wedge a$
 $a \vee b = b \vee a$
 $P \Leftrightarrow Q = Q \Leftrightarrow P$
 $a \oplus b = b \oplus a$ } Commutative Law
- $\uparrow, \downarrow$ } commutative
 $\rightarrow$ } not commutative

- ⑥ $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
 $(a \vee b) \vee c = a \vee (b \vee c)$
 $(a \leftrightarrow b) \leftrightarrow c = a \leftrightarrow (b \leftrightarrow c)$
 $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ } associative laws
- $\uparrow, \downarrow, \rightarrow$ } not associative
- ⑦ $p \wedge p = p$
 $p \vee p = p$ } Idempotent law
- No other operators are idempotent
- ⑧ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
 $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ } Distributive law
- ✓ $\vdash \Rightarrow$ Implication is distributive over conjunction
- ie. $p \rightarrow (q \wedge r) = (p \rightarrow q) \wedge (p \rightarrow r)$
- $\bar{p} + qr$ $(\bar{p} + q)(\bar{p} + r)$
 $= \bar{p} + qr$

$$\begin{aligned} \textcircled{9} \quad \neg(P \wedge Q) &= \neg P \vee \neg Q \\ \neg(P \vee Q) &= \neg P \wedge \neg Q \end{aligned} \quad \left. \begin{array}{l} \overline{AB} = \overline{A+B} \\ \overline{(A+B)} = \overline{A} \overline{B} \end{array} \right\} \text{De Morgan's Law}$$

→ in any big expression to find the complement change $P \rightarrow \bar{P}$
 $\bar{P} \rightarrow P$

$$\text{eg: } C: (P \wedge \bar{Q}) \vee (\bar{P} \vee Q)$$

$$\bar{C}: (\bar{P} \vee \bar{Q}) \wedge (P \wedge \bar{Q})$$

$$\begin{aligned} \wedge &\rightarrow \vee \\ \vee &\rightarrow \wedge \end{aligned}$$

$$\textcircled{10} \quad \begin{aligned} P + PQ &= P \\ P(P+Q) &= P \end{aligned} \quad \left. \right\} \text{Absorption Law}$$

$$\begin{aligned} P + \bar{P}Q &= \underline{P + PQ} + \bar{P}Q \\ &= P + Q(P + \bar{P}) \\ &= P + Q \end{aligned} \quad \left. \right\} \text{Using Absorption in reverse}$$

$$P(\bar{P} + Q) = PQ$$

$$\textcircled{11} \quad \text{Implication Law} \quad \left. \begin{array}{l} P \rightarrow Q = \bar{P} + Q \\ Q' \rightarrow P' \end{array} \right\} \star \star \star$$

$$\textcircled{12} \quad \text{Bi Implication} \quad \begin{aligned} P \leftrightarrow Q &= PQ + \bar{P}\bar{Q} \\ Q &\leftrightarrow P \end{aligned}$$

Simplify & prove that false/true

$$\textcircled{1} \quad \neg P \wedge (P \vee Q) \rightarrow Q \quad \equiv \quad P + (\bar{P}\bar{Q}) + Q \quad = \quad \bar{P} + \bar{Q} + Q \quad = \text{true}$$

$$\textcircled{2} \quad (P \rightarrow Q) \vee P \quad \equiv \quad \bar{P} + Q + P \quad = \text{true}$$

$$\textcircled{3} \quad (P \wedge Q) \rightarrow (P \vee R) \quad \equiv \quad \bar{P} + Q + P + R \quad = \text{true}$$

$$\textcircled{4} \quad P \rightarrow (Q \rightarrow P) \quad \equiv \quad \bar{P} + \bar{Q} + P \quad = \text{true}$$

$$\textcircled{5} \quad \neg r \vee (\neg r \rightarrow p) \quad \equiv \quad \bar{r} + r + p \quad = \text{true}$$

(1 pt) Match the following propositions with their types below.

1. $(p \wedge q) \rightarrow \neg p$
2. $(p \wedge q) \vee (q \leftrightarrow p)$
3. $\neg p \wedge \neg(p \rightarrow q)$
4. $(p \vee \neg p) \rightarrow (p \wedge \neg p)$

- A. Tautology
B. Contradiction
C. Contingency

- 1) $\overline{p} + \overline{q} + \overline{p} = \overline{p} + \overline{q}$ } Contingency
2) $pq + qp + \overline{q}\overline{p}$ } Contingency
3) $\overline{p}(\overline{p} + q) \equiv \overline{p}(p\overline{q}) \rightarrow$ Contradiction
4) $\overline{p + \overline{p}} + p\overline{p} \rightarrow$ Contradiction

$$(q \rightarrow p) \vee (\neg q \rightarrow \neg p) \equiv \overline{q} + p + q + \overline{p} \rightarrow \text{true always}$$

$(\overline{p} \vee q) \wedge (p \rightarrow q) \wedge (\neg q) \rightarrow q$ is tautology or not.

$$\equiv (\overline{p}\overline{q}) + p\overline{q} + q + q$$

$$\equiv p + q + q + \overline{p}\overline{q}$$

$$\equiv p + q + \overline{p}\overline{q} \rightarrow \text{true always}$$

Logical arguments

eg:- $p \rightarrow q$: if it is raining, He'll take umbrella

- It is raining — Can we infer "He'll take umbrella" — Yes
- It is not raining — Can we infer "He'll not take umbrella" — No
- Can we infer "He'll take umbrella" — No

Premise
Knowledge Base
Antecedents
Hypothesis



Conclusion
Consequences

Definition: An argument has the form

A_1
 A_2
 $\vdots$
 A_n
—
 B

$A_1, \dots, A_n$ are called the *premises* of the argument; B is called the *conclusion*. An argument is *valid* if, whenever the premises are true, then the conclusion is true.

argument

- Sequence of statements that derives conclusion from premise
- Valid if given premise is true, then conclusion must be true.
- Invalid if it is possible for all the premise to be true, still conclusion is false.

Argument $\left. \begin{array}{c} A \\ B \\ C \\ \hline D \end{array} \right\}$ is valid if $A \wedge B \wedge C \rightarrow D$ — is a tautology
is invalid if $A \wedge B \wedge C \rightarrow D$ — is not a tautology

Methodology to determine Valid Argument $\rightarrow$

- ① Make the conclusion (D) false
- ② Try to make all the premise (A, B, C) true
- ③ If it is possible $\Rightarrow$ Arg. Invalid
else $\Rightarrow$ Arg. Valid

eg. $\left. \begin{array}{c} P \rightarrow Q \\ P \\ \hline Q \end{array} \right\}$ valid since if $Q = \text{false}$
the $P \rightarrow Q$ & P cannot be simultaneously be made true.

eg. $\left. \begin{array}{c} P \vee Q \\ P \\ \hline Q \end{array} \right\}$ Invalid since if $Q = \text{false}$
but $P \vee Q$ & P can be made true by $P = \text{true}$

eg. $\left. \begin{array}{c} P \vee Q \\ \neg P \\ \hline Q \end{array} \right\}$ valid $\left| \begin{array}{l} Q = \text{false} \\ \neg P = \text{true} \\ P = \text{false} \\ P \vee Q = \text{true} \rightarrow \text{contradiction} \end{array} \right.$

eg. $\left. \begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline P \rightarrow R \end{array} \right\}$ $P \rightarrow R$ is false
 $\Rightarrow P = \text{true}, R = \text{false}$
 $Q \rightarrow R$ is true
 $\Rightarrow Q = \text{false}$
 $P \rightarrow Q$ is true $\rightarrow$ contradiction
 $\equiv$ Valid argument

by logical analysis

$P \rightarrow Q \equiv \bar{P} + Q$
 $Q \rightarrow R \equiv \bar{Q} + R$
 $P \rightarrow R$
 $\Rightarrow$ if $P=1$ then $R=1$

Q: Is this argument valid?

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow.
Therefore, if it rains today, then we will have a barbecue tomorrow.

It rains today $\rightarrow \neg(\text{Will have barbecue today})$
 $\neg(\text{Will have barbecue today}) \rightarrow \text{Will have barbecue tomorrow}$

$\left. \begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline P \rightarrow R \end{array} \right\}$ Valid argument above question

It rains today $\rightarrow$ We will have barbecue tomorrow

Rules of Inference

→ popular & frequently used Valid argument forms.

- ①
$$\frac{P \rightarrow Q}{P} \quad Q$$
 } Modus Ponens
- ②
$$\frac{P \rightarrow Q}{\neg Q} \quad \neg P$$
 } Modus Tollens
- ③
$$\frac{P}{Q} \quad P$$
 } Conjunctive Simplification
- ④
$$\frac{P \vee Q}{\neg P} \quad Q$$
 } disjunctive Syllogism
- ⑤
$$\frac{A \rightarrow B}{B \rightarrow C} \quad A \rightarrow C$$
 } Transitivity of Implication
≡ Hypothetical syllogism
- ⑥
$$\frac{P}{P \vee Q}$$
 } Addition
- ⑦
$$\frac{P \wedge Q}{P}$$
 } Conjunctive Simplification (Alternate Version)
- ⑧
$$\frac{P \vee Q \quad \neg P \vee R}{Q \vee R} \quad \neg Q \rightarrow R \quad \neg R \rightarrow Q$$
 } Resolution

Example 1. Identify the rules of inference used in each of the following arguments.

(a) Alice is a math major. Therefore, Alice is either a math major or a c.s. major.

→ Addition

(b) If it snows today, the college will close. The college is not closed today. Therefore it did not snow today.

→ Modus Tollens

(c) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will get sunburn.

→ Hypothetical syllogism

Example 2. Use rule of inference to show that the premises "Henry works hard", "If Henry works hard then he is a dull boy", and "If Henry is a dull boy then he will not get the job" imply the conclusion "Henry will not get the job."

$$\begin{array}{l} H \\ H \rightarrow D \\ D \rightarrow \neg J \\ \hline \neg J \end{array} \quad \begin{array}{l} \equiv \\ H \\ H \rightarrow \neg J \\ \neg J \\ \hline \end{array} \quad \begin{array}{l} \text{hypothetical} \\ \text{syllogism} \end{array} \quad \begin{array}{l} \text{Modus} \\ \text{ponens} \end{array}$$

The Inference Symbol ($\models$ or $\vdash$) Entailment $KB \models Q$

→ KB (Knowledge Base / Premise)

$$\begin{array}{l} KB \models Y \\ \text{or } KB \vdash Y \end{array} \quad \begin{array}{l} \vdash \\ \text{or } \vdash \end{array} \quad \frac{KB}{Y} \quad \text{is a valid argument}$$

→ Logically infers

$$\begin{array}{l} \rightarrow \text{if } \frac{P_1}{P_2} \quad P_3 \\ \hline C \end{array} \quad \begin{array}{l} \bullet \text{ is a valid argument} \\ \bullet \text{ i.e. } P_1 \wedge P_2 \wedge P_3 \rightarrow C \\ \bullet \text{ is a tautology} \end{array} \quad \left. \begin{array}{l} \text{then we say } P_1, P_2, P_3 \\ \text{logically infers } C \\ P_1, P_2, P_3 \models C \end{array} \right\}$$

- **Entailment: $KB \models Q$**

- Q is entailed by KB (a set of premises or assumptions) if and only if there is no logically possible world in which Q is false while all the premises in KB are true.
- Or, stated positively, Q is entailed by KB if and only if the conclusion is true in every logically possible world in which all the premises in KB are true.

Logical Consequence

Logical inference

Logical implication

Logical Entailment

Same

• $X \models Y$ if $X \rightarrow Y$ is a tautology

Q:

$KB = \{p \rightarrow q, p, s \rightarrow r\}$

Which of p, q, r, s are inferred by (entailed by) KB?

- $p \rightarrow q, p, s \rightarrow r \models p$ ✓
- $p \rightarrow q, p, s \rightarrow r \models q$ ✓
- $p \rightarrow q, p, s \rightarrow r \models r$
- $p \rightarrow q, p, s \rightarrow r \models s$

p —

p

$s \rightarrow r$

q

First Order Logic aka. Predicate logic

- ⊖ Arguments / Logic that cannot be derived from premise using only propositional logic
- eg: $\rightarrow$ All men are mortal. Socrates is a man } $\rightarrow$ Logically valid
 $\rightarrow$ Socrates is mortal. } $\rightarrow$ proposition P.O.V $\rightarrow$ invalid

First Order Requirements

- ① A variables which can take values from a defined set of values.
- ② We want to talk about objects & their property.
- ③ Relationship btw objects (properties)
- ④ Transformation of object / Functions of objects. eg: $f(x) = x^2$, $f(a) = 4$
 $\text{father}(x)$, $\text{father}(\text{Jesus}) = \text{Joseph}$
- ⑤ Quantification: We want to be able to talk about multiple objects

- ① Domain of Variable $\rightarrow$ Set of all values, that the variable can take
 $\rightarrow$ Every variable has domain
 $\rightarrow$ Propositional variables have domain = $\{\text{True}, \text{False}\}$ OR $\{0, 1\}$

② Property of object

☆ aka [relation
predicate]

eg: $\text{Sum}(x, y, z); x+y=z$

$\rightarrow$ Predicate involving only one object is called a property.

eg: $\text{Vowel}(y)$: y is vowel
 $\text{Vowel}(a)$: True
 $\text{Vowel}(b)$: False

eg: $\text{Even}(x)$: x is even
 $\text{Even}(2)$: True
 $\text{Even}(3)$: False

$P(x, y)$: $x < y$
 $P(1, 3)$: True
 $P(3, 1)$: False
 $P(1, 1)$: False

$\rightarrow$ Property of object is a proposition about the object which takes values $\in \{\text{true}, \text{false}\}$ after assigning a value of object from its domain

$\rightarrow$ Predicate is a statement about multiple variables that may/maynot belong to same domain.

$\rightarrow$ Property = Unary predicate

$\rightarrow$ Once we replace every variable in a predicate with a value from its domain, then predicate becomes a proposition

Predicate

x_1
x_2
x_3
x_4
$\vdots$
x_n

Proposition

$x_1 = \text{value}_1$
$x_2 = \text{value}_2$
$x_3 \vdots$
$x_4 \vdots$
$\vdots$
$x_n = \text{value}_n$

$\in \{\text{True}, \text{False}\}$

$\rightarrow$ Predicate with no variable is already a proposition

$\rightarrow$ In a N variable predicate we need to assign value for N variables to make it proposition.

Quantification

In english $\rightarrow$ few, many, some of, all, every

In predicate $\rightarrow \forall$ (for all) $\mid \exists$ (there exist at least one)

$\rightarrow$ all
Universal Quantifier

$\rightarrow$ some
Existential Quantifier

eg:- all students in college kulu kulu are young.

$\forall x \in \text{College kulu kulu} \text{ Young}(x)$

Universal $\rightarrow \forall P(x) \equiv P(a) \wedge P(b) \wedge P(c) \dots$ all elements in domain

$\rightarrow$ One counter example is sufficient to make it false

Existential $\rightarrow \exists P(x) \equiv P(a) \vee P(b) \vee P(c) \dots$ all elements in domain

$\rightarrow$ One example is sufficient to make it true

$$\begin{aligned}\forall P(x) &\equiv \neg \exists \neg P(x) \\ \forall \neg P(x) &\equiv \neg \exists P(x) \\ \exists P(x) &\equiv \neg \forall \neg P(x) \\ \exists \neg P(x) &\equiv \neg \forall P(x)\end{aligned}$$

- Functions can take different numbers of arguments: if x and y are natural numbers, it makes sense to write $\text{mul}(x, y)$ and $\text{square}(x)$. So mul takes two arguments, and square takes only one.
- Predicates and relations can also be understood in these terms. The predicates $\text{even}(x)$ and $\text{prime}(x)$ take one argument, while the binary relations $\text{divides}(x, y)$ and $\text{lt}(x, y)$ take two arguments.
- Functions are different from predicates! A function takes one or more arguments, and returns a *value*. A predicate takes one or more arguments, and is either true or false. We can think of predicates as returning propositions, rather than values.

$\rightarrow$ Quantifiers applied on a predicate turns one of the free variable into a bounded variable

$\rightarrow$ If a predicate has no free variable, then it is a proposition

13. Determine the truth value of each of these statements if the domain consists of all integers.

- a) $\forall n(n+1 > n)$ **T** b) $\exists n(2n = 3n)$ **T**
c) $\exists n(n = -n)$ **T** d) $\forall n(3n \leq 4n)$ **F**

Tricky Cases in Quantifiers

- ① Domain Empty. $\rightarrow$ Domain non-empty unless explicitly given
- $\rightarrow \forall x P(x) \rightarrow \text{True}$, since no counter example
- $\exists x P(x) \rightarrow \text{False}$, since there is no witness

Universally-quantified statements are said to be **vacuously true** in empty worlds.

Existentially-quantified statements are false in an empty world, since nothing exists, period!

② English Translation

- a) Every animal is cute $\forall x \in \text{Animal} \text{ Cute}(x)$
- b) Some animals are cute $\exists x \in \text{Animal} \text{ Cute}(x)$
- c) Every rabbit is cute $\forall x \in \text{animal} [\text{Rabbit}(x) \rightarrow \text{Cute}(x)]$
- d) Some rabbits are cute $\exists x \in \text{animal} [\text{Rabbit}(x) \wedge \text{Cute}(x)]$

Domain = set of all animals

- e) Every happy person wears a hat $\forall x [\text{happy}(x) \rightarrow \text{hat}(x)]$
 f) Some happy person wears a hat $\exists x [\text{happy}(x) \wedge \text{hat}(x)]$

All P's are Q's	$\xrightarrow{\text{translate to}}$	$\forall x [P(x) \rightarrow Q(x)]$
Some P is a Q	$\xrightarrow{\hspace{1cm}}$	$\exists x [P(x) \wedge Q(x)]$
No P's are Q's	$\xrightarrow{\hspace{1cm}}$	$\forall x [P(x) \rightarrow \neg Q(x)]$
Some P's aren't Q's	$\xrightarrow{\hspace{1cm}}$	$\exists x [P(x) \wedge \neg Q(x)]$
Only P's are Q's	$\xrightarrow{\hspace{1cm}}$	$\forall x [Q(x) \rightarrow P(x)] \equiv \forall x [\neg P(x) \rightarrow \neg Q(x)]$
All & only P's are Q's	$\xrightarrow{\hspace{1cm}}$	$\forall x [P(x) \leftrightarrow Q(x)] \equiv \forall x [Q(x) \leftrightarrow P(x)]$

☆ $\forall_{P(x)} Q(x) \equiv \forall x [P(x) \rightarrow Q(x)]$
 $\exists_{P(x)} Q(x) \equiv \exists x [P(x) \wedge Q(x)]$

8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

- a) $\forall x (R(x) \rightarrow H(x))$ b) $\forall x (R(x) \wedge H(x))$
 c) $\exists x (R(x) \rightarrow H(x))$ d) $\exists x (R(x) \wedge H(x))$

- a) All rabbits hop
 b) All animals are rabbits & hops
 c) There exists an animal which either not a rabbit or it hops
 d) There exists an animal which is both a rabbit and it hops $\equiv$ Some rabbits hop

- No P's are Q's $\equiv \forall x (P(x) \rightarrow \neg Q(x)) \equiv \neg \exists x (P(x) \wedge Q(x))$
- No rabbit is cute $\equiv$ All rabbits are noncute $\equiv$ There doesn't exist a animal which is both a rabbit & is cute
- Only males are in the army $\equiv \forall x [\text{Army}(x) \rightarrow \text{Male}(x)]$
- All & only males are in the army $\equiv \forall x [\text{Army}(x) \leftrightarrow \text{Male}(x)]$
- Every orange cat is fluffy $\equiv \forall x [\text{OrangeCat}(x) \rightarrow \text{fluffy}(x)] \equiv \forall x [\text{Orange}(x) \wedge \text{Cat}(x) \rightarrow \text{fluffy}(x)]$
- Some orange cat is fluffy $\equiv \exists x (\text{Orange}(x) \wedge \text{Cat}(x) \wedge \text{fluffy}(x))$
- Only orange cats are fluffy $\equiv \forall x [\text{fluffy}(x) \rightarrow \text{Orange}(x) \wedge \text{Cat}(x)]$
- For any natural numbers n
 n is even if & only if
 n^2 is even $\left. \vphantom{\begin{matrix} n \\ n^2 \end{matrix}} \right\} \equiv \forall n \in \mathbb{N} [\text{Even}(n) \leftrightarrow \text{Even}(n^2)]$
 $\equiv \forall n [\text{natural}(n) \rightarrow (\text{Even}(n) \leftrightarrow \text{Even}(n^2))]$

What do the statements $\forall x < 0 (x^2 > 0)$, $\forall y \neq 0 (y^3 \neq 0)$, and $\exists z > 0 (z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

→ For all negative real number x , x^2 is positive

→ For all non zero real numbers x , x^3 is non zero

→ There exists a positive real number z such that $z^2 = 2$

Bounded & Unbounded Variable

Quantified Variable

eg:- All natural numbers are even $\equiv \forall x \in \mathbb{N} E(x) \equiv E(1) \wedge E(2) \wedge E(3) \dots$

- Quantify a free variable becomes → One less free variable
- Quantify a predicate

• if no more free variable, then it becomes a proposition

→ Quantified variable are not free

→ They are bounded to values within the domain

→ It can be said even that they are not really variable.

→ This is not a predicate but a proposition, since there is no free variable

Free Variable

$P(m, n): m+n > 4$
Domain is set of Integers

- $P(1, 2) \rightarrow$ proposition
- $P(0, n) \rightarrow$ predicate (n is a free variable)
- $\forall n P(0, n) \rightarrow$ proposition (n is a bounded variable)

★ No Free Variable $\Rightarrow$ proposition

if expression A is already a proposition (ie no free variable)
then $\forall x A \equiv A$; $\exists x A \equiv A$

Scope of a Quantified Variable

→ part of the expression to which quantifier is applied.
→ $\forall x [\dots]$
scope of $\forall x$

$\exists y [P_y \vee H_x]$

$\exists y [F_y \wedge \forall x [F_x \rightarrow G_{xy}]]$

$\exists y [F_y \wedge \forall x [F_x]] \wedge L_y$

→ Scope changes meaning.

eg. $\forall x (\text{Smile}(x) \rightarrow \text{Wearhat})$

$\forall x \text{Smile}(x) \rightarrow \forall x \text{Wearhat}(x)$

eg. $\exists x (\text{Smile}(x) \wedge \text{Wearhat}(x))$

$\exists x \text{Smile}(x) \wedge \exists x \text{Wearhat}(x)$

Nested Quantifiers

↳ one quantifier within the scope of another

eg. $\forall x \exists y (x+y=0)$

eg. No even numbers greater than 2 is prime $\equiv \forall x (Even(x) \wedge (x > 2) \rightarrow \neg prime(x))$
 $\equiv \neg \exists x (Even(x) \wedge (x > 2) \wedge prime(x))$

eg. Let $P(x,y): x+y=17$, Now find truth value of domain $\Rightarrow$ set of integers ($\mathbb{I}$)

- a) $\forall x \forall y P(x,y)$ F
- b) $\forall x \exists y P(x,y)$ T
- ☆ c) $\exists x \forall y P(x,y)$ F
- d) $\exists x \exists y P(x,y)$ T

$$\exists x \forall y \neq \forall y \exists x$$

There exist a x such that for all y $\neq$ For all y , there exist a x

eg. $T(x,y) \rightarrow$ student x has taken course y .

Domain (x) $\rightarrow \{a, b, c\}$

Domain (y) $\rightarrow \{1, 2, 3, 4\}$

- $\forall x \forall y T(x,y) \rightarrow$ All students has taken all course
- $\forall x \exists y T(x,y) \rightarrow$ All students has taken atleast one course
- $\exists x \forall y T(x,y) \rightarrow$ Atleast one student has taken all courses
- $\exists x \exists y T(x,y) \rightarrow$ Atleast one student has taken a course
- $\forall y \exists x T(x,y) \rightarrow$ All courses are taken by some student
- $\exists y \forall x T(x,y) \rightarrow$ Atleast one course is taken by all students

$\forall x \exists y \neq \exists x \forall y$	$\forall x \forall y \equiv \forall y \forall x$
$\forall x \exists y \neq \exists y \forall x$	$\exists x \exists y \equiv \exists y \exists x$



1) Determine the truth value of each expression below if the domain is the set of all real numbers.

- a) $\exists x \forall y (xy = 0)$ (If true, give an example.)
- b) $\forall x \forall y \exists z (z = (x-y)/3)$ (If false, give a counterexample.)
- c) $\forall x \forall y (xy = yx)$ (If false, give a counterexample.)
- d) $\exists x \exists y \exists z (x^2 + y^2 = z^2)$ (If true, give an example.)

True $x=0$

True

True

$$3^2 + 4^2 = 5^2 \text{ (True)}$$

THINKING OF QUANTIFICATION AS LOOPS In working with quantifications of more than one variable, it is sometimes helpful to think in terms of nested loops. (Of course, if there are infinitely many elements in the domain of some variable, we cannot actually loop through all values. Nevertheless, this way of thinking is helpful in understanding nested quantifiers.) For example, to see whether $\forall x \forall y P(x, y)$ is true, we loop through the values for x , and for each x we loop through the values for y . If we find that $P(x, y)$ is true for all values for x and y , we have determined that $\forall x \forall y P(x, y)$ is true. If we ever hit a value x for which we hit a value y for which $P(x, y)$ is false, we have shown that $\forall x \forall y P(x, y)$ is false.

Similarly, to determine whether $\forall x \exists y P(x, y)$ is true, we loop through the values for x . For each x we loop through the values for y until we find a y for which $P(x, y)$ is true. If for every x we hit such a y , then $\forall x \exists y P(x, y)$ is true; if for some x we never hit such a y , then $\forall x \exists y P(x, y)$ is false.

To see whether $\exists x \forall y P(x, y)$ is true, we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y . Once we find such an x , we know that $\exists x \forall y P(x, y)$ is true. If we never hit such an x , then we know that $\exists x \forall y P(x, y)$ is false.

Finally, to see whether $\exists x \exists y P(x, y)$ is true, we loop through the values for x , where for each x we loop through the values for y until we hit an x for which we hit a y for which $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false only if we never hit an x for which we hit a y such that $P(x, y)$ is true.

Numerical Quantification

1) Atleast two $\exists x \exists y (x \neq y \wedge P(x) \wedge P(y))$

2) Atmost two $\neg \exists x \exists y \exists z (x \neq y \wedge y \neq z \wedge x \neq z \wedge P(x) \wedge P(y) \wedge P(z))$
 $\equiv \forall x \forall y \forall z [Cube(x) \wedge Cube(y) \wedge Cube(z) \rightarrow (x=y \vee y=z \vee x=z)]$

3) **Exactly one** $\exists x \forall y [x=y \vee (P(x) \wedge \neg P(y))] \equiv \exists x \forall y [x \neq y \rightarrow P(x) \wedge \neg P(y)]$

4) Atleast three $\exists x \exists y \exists z [x \neq y \wedge y \neq z \wedge x \neq z \wedge P(x) \wedge P(y) \wedge P(z)]$

5) Atmost one $\neg \exists x \exists y [x \neq y \wedge P(x) \wedge P(y)] \equiv \forall x \forall y [P(x) \wedge P(y) \rightarrow x=y]$

6) Exactly two $\exists x \exists y \forall z [(x \neq z \wedge y \neq z) \rightarrow P(x) \wedge P(y) \wedge \neg P(z)]$

$\rightarrow \exists x [P(x) \wedge \forall y (P(y) \rightarrow x=y)] \equiv \exists x \forall y [P(x) \wedge (P(y) \rightarrow x=y)]$

☆ $\equiv \exists x \forall y (P(y) \leftrightarrow x=y) \neq \exists x \forall y (P(x) \leftrightarrow x=y)$

Practice:

1) for every number there is a larger number

$$\forall x \exists y (y > x)$$

2) There is a number that is larger than every other number

$$\exists x \forall y (x \neq y \rightarrow x > y)$$

3) Every positive number is a square

$$\forall x (x > 0 \rightarrow \exists z (z^2 = x))$$

4) if one number is less than another, then there is a number, then, there is a number properly between the two

$$\forall x \forall y [x < y \rightarrow \exists z (x < z \wedge z < y)]$$

5) There are atleast two numbers which are prime

$$\exists x \exists y [x \neq y \wedge \text{prime}(x) \wedge \text{prime}(y)]$$

6) There are atleast three numbers which are prime

$$\exists x \exists y \exists z [x \neq y \wedge y \neq z \wedge x \neq z \wedge \text{prime}(x) \wedge \text{prime}(y) \wedge \text{prime}(z)]$$

7) ☆ There are infinitely many numbers which are prime $\forall x \exists y [(y > x) \wedge \text{prime}(y)]$

8) There is atmost one number which is prime $\forall x \forall y [\text{prime}(x) \wedge \text{prime}(y) \rightarrow x=y]$

9) There is exactly one # which is prime $\exists x \forall y [\text{prime}(y) \leftrightarrow x=y]$
 $\exists x [\text{prime}(x) \wedge \forall y [\text{prime}(y) \rightarrow x=y]]$

10) x is a prime

$$\forall y (x \cdot y = 0 \rightarrow y = 1 \vee y = x)$$

11) There is an infinite number of pairs of prime that differ by number 2. [Twin Prime Conjecture]

☆ $\forall x \exists y [y > x \wedge \text{prime}(y+2) \wedge \text{prime}(y)]$

12) All even numbers greater than two are the sum of two primes [Goldbach's Conjecture]

$$\forall x [x > 2 \wedge \text{even}(x) \rightarrow \exists y [\text{prime}(y) \wedge \text{prime}(x-y) \wedge y > 0]]$$

Consider this formula

$$\Phi = \forall y. ((P(y) \wedge Q(x)) \vee \forall x. (P(x) \vee Q(y) \wedge P(z)))$$

How many free variables does this formula contain? Clicker

- ☐ One
- ☒ Two
- ☐ Three
- ☐ Four
- ☐ None

17. Translate into English:

- (a) $\forall x (E(x) \rightarrow E(x+2))$. $\rightarrow$ 2 more than an even number is also even
- (b) $\forall x \exists y (\sin(x) = y)$. $\rightarrow$ $\sin x$ is always a real number for real x
- (c) $\forall y \exists x (\sin(x) = y)$. $\rightarrow$ For all real y , there exists x such that $\sin(x) = y$
- (d) $\forall x \forall y (x^3 = y^3 \rightarrow x = y)$. $\rightarrow$ For all real numbers have unique cube root

20. Consider the statement, "For all natural numbers n , if n is prime, then n is solitary." You do not need to know what *solitary* means for this problem, just that it is a property that some numbers have and others do not.

- (a) Write the converse and the contrapositive of the statement, saying which is which. Note: the original statement claims that an implication is true for all n , and it is that implication that we are taking the converse and contrapositive of.

original $\forall n \in \mathbb{N} (\text{prime}(n) \rightarrow \text{solitary}(n))$

converse $\forall n \in \mathbb{N} (\text{solitary}(n) \rightarrow \text{prime}(n))$

contrapositive $\forall n \in \mathbb{N} (\neg \text{solitary}(n) \rightarrow \neg \text{prime}(n))$

Q5 Let $P(x)$, $Q(x)$, $R(x)$, $S(x)$ and $T(x, y)$ denote the following predicates with domain $\mathbb{Z}$:

$P(x): x^2 = x$,

$Q(x): x \leq 0$,

$R(x): x^2 = x + 1$,

$S(x): x$ is even,

$T(x, y): (x < y) \wedge (y < x^2)$

Determine whether each of the following statements is true or false, and give brief reasons.

- (a) $\forall x \in \mathbb{Z}, P(x) \rightarrow Q(x)$ False: $P(1) \rightarrow Q(1)$
- (b) $\forall x \in \mathbb{Z}, P(x) \rightarrow \neg Q(x)$ False: $P(0) \rightarrow \neg Q(0)$
- (c) $\forall x \in \mathbb{Z}, R(x) \rightarrow P(x)$ True
- (d) $\forall x \in \mathbb{Z}, P(x) \rightarrow R(x)$ False
- (e) $\forall x \in \mathbb{Z}, (P(x) \wedge S(x)) \rightarrow Q(x)$ True
- (f) $\forall x \in \mathbb{Z}, (P(x) \wedge Q(x)) \rightarrow S(x)$ True
- (g) $\exists x \in \mathbb{Z}$ such that $R(x)$ False
- (h) $\exists x \in \mathbb{Z}$ such that $S(x) \wedge Q(x)$ True
- (i) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $T(x, y)$ False $\rightarrow T(0, y) \text{ \& } T(1, y) \rightarrow \text{conclude}$
- (j) $\forall x \in \mathbb{Z}, (\neg P(x) \rightarrow \exists y \in \mathbb{Z} \text{ such that } T(x, y))$ True

$S(0) = \text{True}$

$R(x) \Rightarrow$

$x^2 - x - 1 = 0$

$\frac{1 \pm \sqrt{1+4}}{2}$

$= \frac{1 \pm \sqrt{5}}{2}$

not in $\mathbb{Z}$

$P(x)$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

integer

Negation of Quantifiers

$$\begin{array}{l|l} \textcircled{1} S: \text{Everyone has a car} & \text{Negation?} \\ S: \forall x \text{ Car}(x) & \neg S: \text{Someone doesn't have a car} \\ & \neg S: \exists x (\neg \text{Car}(x)) \end{array}$$

$$\begin{array}{l|l} \textcircled{2} M: \text{At least one person has a car} & \neg M: \text{Nobody has a car} \\ M: \exists x \text{ Car}(x) & \neg M: \forall x \neg \text{Car}(x) \end{array}$$

$$\begin{array}{l} \neg \forall P(x) \equiv \exists x \neg P(x) \\ \neg \exists P(x) \equiv \forall x \neg P(x) \end{array}$$

$$\begin{array}{l} \neg \forall \equiv \exists \neg \\ \neg \exists \equiv \forall \neg \end{array} \quad \left. \begin{array}{l} \text{quantifier flips} \\ \text{Negation goes inside} \end{array} \right\}$$

$$\forall x P(x) \quad P(x_1) \wedge P(x_2) \wedge P(x_3) \dots$$

$$\exists x P(x) \quad P(x_1) \vee P(x_2) \vee P(x_3) \dots$$

↑ Above result is just De Morgan's Law

$$\begin{aligned} \text{eg1} \quad & \neg \forall x \exists y [P(x) \rightarrow Q(y)] \\ & \equiv \exists x \neg \exists y [P(x) \rightarrow Q(y)] \\ & \equiv \exists x \forall y [P(x) \wedge \neg Q(y)] \end{aligned}$$

$$\begin{array}{l} \neg (P \rightarrow Q) \\ \equiv P \wedge \bar{Q} \end{array}$$

$$\begin{array}{l} \neg (P \wedge Q) \\ \equiv P \rightarrow \bar{Q} \end{array}$$

eg2 Negation of "There is a cute puppy"

$$\neg \exists x (\text{Puppy}(x) \wedge \text{Cute}(x)) \equiv \forall x [\text{Puppy}(x) \rightarrow \neg \text{Cute}(x)]$$

eg3 Every cat is cute :S

$$\begin{aligned} \neg S: \quad & \neg \forall x (\text{Cat}(x) \rightarrow \text{Cute}(x)) \\ & \therefore \exists x (\text{Cat}(x) \wedge \neg \text{Cute}(x)) \end{aligned}$$

VALIDITY & SATISFIABILITY of FOL

Valid $\rightarrow$ Always true $\rightarrow$ Given whatever nonempty domain & predicate
 Satisfiable $\rightarrow$ Sometimes true $\rightarrow$ For some domain or predicate

eg. $\forall x P(x) \rightarrow$ satisfiable $\left| \begin{array}{l} x \in \mathbb{N} \\ P(x) = x \geq 1 \end{array} \right.$

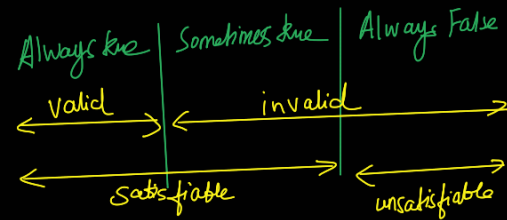
eg. $\exists x P(x) \rightarrow$ Invalid, satisfiable

eg. $\forall x \exists y P(x, y) \rightarrow$ Invalid, satisfiable

eg. $S: \exists x P(x) \vee \exists y \neg P(y) \rightarrow$ Valid (for non empty domain)
 Satisfiable

eg. $\forall x \exists y (x+y=0) \rightarrow$ Satisfiable, Invalid

eg. $\forall x P(x) \wedge \exists y \neg P(y) \rightarrow$ Invalid, unsatisfiable



1. In a FOL Expression, by default, **ALL Variables have Same Domain.** ★

2. Domain in FOL, is ALWAYS Non-Empty..
 (unless explicitly gives as empty)

eg:- $S: \forall x P(x) \rightarrow \exists x P(x)$ $\rightarrow$ Valid
 $\rightarrow$ Satisfiable

★ $\rightarrow$ Assume domain empty $\rightarrow$ unsatisfiable

Validity & Implication

$\forall x P(x) \rightarrow \exists x P(x) \rightarrow$ Valid

$\alpha \rightarrow \beta \rightarrow$ Method 1 $\rightarrow$ Assume α - true
 Try to make β - false

Method 2 $\rightarrow$ Assume β - false
 Try to make α - True

} If not possible $\equiv$ Valid

eg:- $\forall x [P(x) \wedge Q(x)] \rightarrow \forall x P(x) \wedge \forall x Q(x)$

eg: $\forall x P(x) \wedge \forall x Q(x) \rightarrow \forall x [P(x) \wedge Q(x)]$

Valid }
 Valid }

$\forall x [P(x) \wedge Q(x)] \equiv \forall x P(x) \wedge \forall x Q(x)$
 Universal Quantifier distributive over conjunction

$\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$
 Existence Quantifier distributive over disjunction

Logical Equivalence

- if $\alpha \rightarrow \beta$ & $\beta \rightarrow \alpha$ is valid implication involving FOL then α & β are equivalent FOL. (α & β are equivalent FOL if $\alpha \leftrightarrow \beta$ is valid)

eg.

Negation

$$\begin{array}{l|l} \forall x P(x) \equiv \neg \exists x \neg P(x) & \neg \forall x P(x) \equiv \exists x \neg P(x) \\ \exists x P(x) \equiv \neg \forall x \neg P(x) & \neg \exists x P(x) \equiv \forall x \neg P(x) \end{array}$$

Distributive

$$\begin{array}{l} \forall x [P(x) \wedge Q(x)] \equiv \forall x P(x) \wedge \forall x Q(x) \\ \exists x [P(x) \vee Q(x)] \equiv \exists x P(x) \vee \exists x Q(x) \end{array}$$

Unique/Exactly one

$$\exists x [P(x) \wedge \forall y (P(y) \rightarrow x=y)] \equiv \exists x \forall y [P(y) \leftrightarrow x=y]$$

Commutism

$$\begin{array}{l|l} \forall x \exists y \neq \exists x \forall y & \forall x \forall y \equiv \forall y \forall x \\ \exists x \exists y \neq \exists y \forall x & \exists x \exists y \equiv \exists y \exists x \end{array}$$

Domain to Universe

$$\begin{array}{l} \forall_{P(x)} Q(x) \equiv \forall x [P(x) \rightarrow Q(x)] \\ \exists_{P(x)} Q(x) \equiv \exists x [P(x) \wedge Q(x)] \end{array}$$

$$\forall x [P(x) \vee Q(x)] \not\equiv \forall x P(x) \vee \forall x Q(x)$$

$$\exists x [P(x) \wedge Q(x)] \not\equiv \exists x P(x) \wedge \exists x Q(x)$$

$$\exists x [P(x) \oplus Q(x)] \not\equiv \exists x P(x) \oplus \exists x Q(x)$$

$$\forall x [P(x) \oplus Q(x)] \not\equiv \forall x P(x) \oplus \forall x Q(x)$$

$$\forall x [P(x) \rightarrow Q(x)] \not\equiv \forall x P(x) \rightarrow \forall x Q(x)$$

$$\exists x [P(x) \rightarrow Q(x)] \not\equiv \exists x P(x) \rightarrow \exists x Q(x)$$

$$\forall x [P(x) \leftrightarrow Q(x)] \not\equiv \forall x P(x) \leftrightarrow \forall x Q(x)$$

	$\forall$	$\exists$
$\wedge$	$\leftrightarrow$	$\rightarrow$
$\vee$	$\leftarrow$	$\leftrightarrow$
$\rightarrow$	$\rightarrow$	$\leftarrow$
$\leftrightarrow$	$\rightarrow$	$\times$
$\oplus$	$\times$	$\leftarrow$
$\uparrow$	$\rightarrow$	$\leftarrow$
$\downarrow$	$\rightarrow$	$\leftarrow$

- $\forall x (P(x) \vee Q(x)) \models \forall x P(x) \vee \forall x Q(x)$ $\times$
- $\forall x (P(x) \rightarrow Q(x)) \models \forall x P(x) \rightarrow \forall x Q(x)$ $\checkmark$
- $\forall x P(x) \rightarrow \forall x Q(x) \models \forall x (P(x) \rightarrow Q(x))$ $\times$
- $\neg \forall x (P(x) \wedge Q(x)) \models \exists x \neg P(x) \wedge \exists x \neg Q(x)$ $\times$
- $\exists x P(x) \wedge \exists x Q(x) \models \exists x (P(x) \wedge Q(x))$ $\times$

$$\begin{aligned} \neg \forall x [P(x) \wedge Q(x)] &\equiv \exists x [\neg P(x) \vee \neg Q(x)] \\ &\equiv \exists x \neg P(x) \vee \exists x \neg Q(x) \end{aligned}$$

Q) $\forall x [P(x) \oplus Q(x)] \not\equiv \exists x P(x) \rightarrow \forall x Q(x)$

not valid in both direction.

domain $\{1, 2, 4, 6, 8\}$

$P(x) : \text{odd}(x)$

$\exists x P(x) \rightarrow T$

$Q(x) : \text{even}(x)$

$\forall x Q(x) \rightarrow F$

$\forall x [\text{odd} \oplus \text{even}] \rightarrow T$

NULL QUANTIFICATION

$$\forall x [P(x) \vee A] \equiv \forall x P(x) \vee \forall x A \equiv \forall x P(x) \vee A$$

$$\exists x [P(x) \vee A] \equiv \exists x P(x) \vee A$$

$$\forall x [P(x) \wedge A] \equiv \forall x P(x) \wedge A$$

$$\exists x [P(x) \wedge A] \equiv \exists x P(x) \wedge A$$

• where A is not a predicate but a proposition

$$\forall x A = A, \exists x A = A$$

• A is unaffected by $\forall/\exists$ even though A is inside scope

$$\forall x [A \rightarrow P(x)] \equiv A \rightarrow \forall x P(x)$$

$$\exists x [A \rightarrow P(x)] \equiv A \rightarrow \exists x P(x)$$

$$\exists x [P(x) \rightarrow A] \equiv \exists x \neg P(x) \vee A \equiv \forall x P(x) \rightarrow A$$

$$\forall x [P(x) \rightarrow A] \equiv \forall x \neg P(x) \vee A \equiv \exists x P(x) \rightarrow A$$

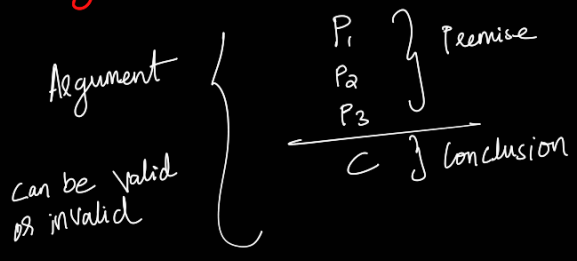
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Which one of the following predicate formulae is NOT logically valid?

33 Note that W is a predicate formula without any free occurrence of x.

A. $\forall x(p(x) \vee W) \equiv \forall x(p(x)) \vee W$
 B. $\exists x(p(x) \wedge W) \equiv \exists x(p(x)) \wedge W$
 C. $\forall x(p(x) \rightarrow W) \equiv \forall x(p(x)) \rightarrow W$ → Not Valid
 D. $\exists x(p(x) \rightarrow W) \equiv \forall x(p(x)) \rightarrow W$

Arguments in FOL



An argument is valid if $P_1 \wedge P_2 \wedge P_3 \rightarrow C$ is valid (always true)

eg: $\frac{\forall x P(x)}{P(a)} \rightarrow \text{valid}$ Universal instantiation

$a \in \text{Domain}$

eg: $\frac{P(b)}{\forall x P(x)} \rightarrow \text{Invalid}$ $b \in \text{domain}$

1) $\frac{\forall x P(x)}{P(a), a \in \text{domain}}$ — Universal instantiation

2) $\frac{\text{for arbitrary } m, P(m)}{\forall x P(x)}$ — Universal generalisation

3) $\frac{P(a), a \in \text{domain}}{\exists x P(x)} \rightarrow \text{Existential generalisation}$

m - arbitrary
 a - specific $a \in \text{domain}$

4) $\frac{\exists x P(x)}{\text{for some particular } m \in \text{domain}, P(m)} \rightarrow \text{Existential instantiation}$

TABLE 2 Rules of Inference for Quantified Statements.	
Rule of Inference	Name
$\frac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

eg:- $\frac{\forall x [P(x) \rightarrow Q(x)]}{P(a)} \rightarrow \text{Valid}$ Similar to Modus Ponens

eg:- $\frac{\forall x [P(x) \rightarrow Q(x)]}{\neg Q(a)} \rightarrow \text{Valid}$ Similar to Modus Tollens

eg:- $\frac{\exists x [P(x) \rightarrow Q(x)]}{P(a)} \rightarrow \text{Invalid}$

(1) Every man is mortal
 Aristotle is a man
 Therefore: Aristotle is mortal

$\frac{\forall x [\text{man}(x) \rightarrow \text{Mortal}(x)]}{\text{Man(Aristotle)}} \rightarrow \text{Valid}$ Modus ponens

(2) Aristotle is a man
 Therefore: Someone is a man

$\frac{\text{Man(Aristotle)}}{\exists x \text{Man}(x)} \rightarrow \text{Valid} \rightarrow \text{Existential generalisation}$

27. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.

$\frac{\forall x [P(x) \rightarrow (Q(x) \wedge S(x))]}{\forall x [P(x) \wedge R(x)]} \rightarrow \forall x [Q(x) \wedge S(x)]$
 $\frac{\forall x [P(x) \wedge R(x)]}{\forall x [R(x) \wedge S(x)]} \equiv \forall x P(x) \wedge \forall x Q(x)$

28. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

$$\begin{array}{l} \forall x [P(x) \vee Q(x)] \\ \forall x [\neg P(x) \wedge Q(x) \rightarrow R(x)] \\ \hline \forall x [\neg R(x) \rightarrow P(x)] \end{array}$$

29. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

$$① + ① \Rightarrow \exists x Q(x) \text{ --- } ②$$

$$a + ② \Rightarrow \exists x S(x) \text{ --- } ③$$

$$\exists \equiv \forall x [\neg R(x) \vee \neg S(x)]$$

$$\exists + ③ \equiv \exists x \neg R(x) \text{ hence valid}$$

$$\begin{array}{l} 1 \quad \forall x [P(x) \vee Q(x)] \rightarrow \text{either } P \text{ or } Q \text{ is always true} \\ 2 \quad \forall x [\neg P(x) \vee S(x)] \rightarrow \text{either } S \text{ true or } Q \text{ false} \\ 3 \quad \forall x [R(x) \rightarrow \neg S(x)] \\ 4 \quad \exists x \neg P(x) \\ \hline \exists x \neg R(x) \end{array}$$